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On the hierarchy of partially invariant submodels of differential equations

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Abstract

It is noted that the partially invariant solution (PIS) of differential equations in many cases can be represented as an invariant reduction of some PISs of the higher rank. This introduces a hierarchic structure in the set of all PISs of a given system of differential equations. An equivalence of the two-step and the direct ways of construction of PISs is proved. The hierarchy simplifies the process of enumeration and analysis of partially invariant submodels to the given system of differential equations. In this framework, the complete classification of regular partially invariant solutions of ideal MHD equations is given.

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Introduction

Symmetry group analysis is the universal tool for construction of exact solutions to a mathematical model written as a system E of differential equations [1, 2]. The base for the symmetry analysis application is the group G of continuous transformations, admitted by system E. The admissible group acts on the set of solutions of E, i.e., it transforms any solution into some other solution. The demands of complete or partial invariance of solution with respect to some subgroup $H \subset G$ provides one with the practical algorithm for construction of exact solutions to system E. H-invariant or partially invariant solutions are described in terms of equations, which are simpler than the original model. They either contain less number of independent variables (for invariant solutions) or split into two subsystems, where one contains less number of independent variables and another involves less unknown functions then the original system (for partially invariant solutions). The system of equations, which determine invariant or partially invariant solutions is referred to as the submodel of the original mathematical model [3]. In some cases, the submodel can be completely integrated and its solution in closed form can be constructed. If the integration cannot be performed, the submodel can be investigated analytically or numerically as an ordinary system of differential equations. The advantage here is that the analysis of simpler equations of the submodel gives exact solutions to the more complicated original model. In particular, the solution of a system with less number of independent variables can be obtained numerically with high accuracy and then used for testing and validating of multidimensional numerical solvers (see [p 434] [4]). Thus, in what follows we do not make a difference between the notions of solution, submodel and reduced system for a system of differential equations.

Construction of exact solution by symmetry analysis methods is the reconnaissance process. One at first obtains a solution and then investigates the properties of the physical process described by the solution. The set of the group-invariant solutions is usually wide enough. Each subgroup of the admissible group is responsible for some (invariant, partially invariant, differentially invariant) solution of the model E. It is known that conjugated subgroups produce equivalent solutions. The complete list of unconjugated subgroups of the symmetry group G is called the optimal system of subgroups of G. It serves as the list of significantly different group-invariant solutions of the system of equations. For the admissible groups of dimensions higher than six optimal systems can be large (hundreds of elements). Investigation of such a big volume of solutions requires its additional classification and regulation. For the invariant solutions one can use a classification scheme provided by LOT lemma [5], which introduces a hierarchic structure on the set of invariants submodels. The lemma states that for any two subgroups $H, N \subset G$, such that H is the normal divisor in N, N-invariant solution to the equations E coincides with N/H-invariant solution to the H-invariant submodel E/H. Note, that H could be the normal divisor in several different subgroups N_i. In this case, the two-step procedure gives an opportunity to inherit information on the first integrals and the properties of solutions to the H-submodel in all of N_i-submodels (see, for example, [6]). A similar result for partially invariant solutions has not been proved yet. The main difficulty here is that partially invariant submodels are defined by overdetermined systems of equations. Investigation of consistency of such systems is a complicated problem, which cannot be traced in general form. In the present paper, it is noted that the overdeterminacy of the partially invariant submodel is not important. From the geometrical point of view, a partially invariant solution forms a manifold in space of functions and independent variables with the property of partial invariance with respect to the group. This is used to prove that under some additional assumptions on subgroups H and N the hierarchy of partially invariant solutions also takes place. For partially invariant solutions this statement is even more valuable as it allows one to perform the compatibility analysis for several partially invariant solutions with respect to different groups N_i by only one compatibility analysis of the higher-rank H-partially invariant submodel.

The proof of the theorem on a hierarchy of partially invariant submodels requires some preliminary information from the general theory of partially invariant solutions. In this connection the paper has the following content. The theory of partially invariant solutions taken from [1] is briefly recounted in sections 1 and 2. The hierarchy of partially invariant submodels is discussed in sections 3 and 4. Here are lemmas 3 and 4 and the main theorem 2 are new. Section 5 is devoted to illustration of the theoretical constructions of previous sections on an example of the hierarchy of partially invariant solution for shallow water equations. The physical interpretation of the solution obtained here can be found in [30]. The hierarchy of non-barochronous regular submodels of ideal MHD equations is constructed in section 6.

1. Partially invariant manifolds

Let $G_r = \{T_a : \bar{x} = f(x, a)\}$, $a \in \Delta \subset \mathbb{R}^r$ be a local Lie group of transformations acting in space $x \in \mathbb{R}^n$. Let the basis of the corresponding Lie algebra L_r of infinitesimal generators be chosen as $X_\alpha = \xi^i_\alpha(x)\partial_{x^i}$, $(i = 1, \ldots, n; \alpha = 1, \ldots, r)$. Hereafter the Einstein summation convention on the repeating upper and lower indices is adopted. Let us observe a manifold $\mathcal{M}$ regularly defined by equations

$$\mathcal{M} : \psi^\sigma(x) = 0, \qquad \sigma = 1, \ldots, s; \qquad \operatorname{rank}\|\partial\psi^\sigma/\partial x^k\| = s. \tag{1.1}$$

Hereafter $\|a^\sigma_k\|$ denotes a matrix with elements a^σ_k; rank $M(x)$ denotes the maximal rank of matrix $M(x)$ for various values of x.

Definition 1. *Orbit of point x under G_r group action is a set of points $\mathcal{O}(x) = \{f(x, a)|a \in \Delta \subset \mathbb{R}^r\}$. Orbit $\mathcal{O}(\mathcal{M})$ of the manifold $\mathcal{M}$ is the locus of orbits of all points $x \in \mathcal{M}$,*

$$\mathcal{O}(\mathcal{M}) = \{f(x, a)|x \in \mathcal{M}, a \in \Delta \subset \mathbb{R}^r\}.$$

Let us introduce the following integer characteristic.

Definition 2. *The defect $\delta(\mathcal{M}, G_r)$ of the manifold $\mathcal{M}$ under G_r group action is a difference between the dimensions of the orbit $\mathcal{O}(\mathcal{M})$ and of the manifold $\mathcal{M}$ itself,*

$$\delta(\mathcal{M}, G_r) = \dim \mathcal{O}(\mathcal{M}) - \dim \mathcal{M}. \tag{1.2}$$

The defect of the manifold is an important characteristic showing the degree of non-invariancy of the manifold $\mathcal{M}$ under the action of G_r.

Definition 3. *Manifold $\mathcal{M}$ is referred to as G_r-invariant manifold if $\delta(\mathcal{M}, G_r) = 0$. Otherwise $\mathcal{M}$ is referred to as G_r-partially invariant manifold with defect $\delta(\mathcal{M}, G_r)$.*

Orbit of G_r-invariant manifold coincide with the manifold. Orbit $\mathcal{O}(\mathcal{M})$ of an arbitrary manifold $\mathcal{M}$ under the group action is itself an invariant manifold of the group because, by definition, orbit of any point belong to the orbit of the manifold. Moreover, orbit $\mathcal{O}(\mathcal{M})$ is the minimal invariant manifold of the group G_r, containing $\mathcal{M}$. Thus, it can be described in terms of the functional relations between the invariants of the group. Let the complete set of functionally independent invariants of G_r be chosen in the form $I = (I^1(x), \ldots, I^t(x))$, where $t = n - r_*$, and $r_* = \operatorname{rank}\|\xi^i_\alpha(x)\|$. By virtue of the theorem of representation of a non-singular invariant manifold [1] the orbit of $\mathcal{M}$ can be written in the form

$$\Phi^\tau(I^1(x), \ldots, I^t(x)) = 0, \qquad \tau = 1, \ldots, l. \tag{1.3}$$

Definition 4. *Under the condition of regularity of specification (1.3), i.e.* $\operatorname{rank}\|\partial\Phi^\tau/\partial I^k\| = l$ *the number*

$$\rho(\mathcal{M}, G_r) = t - l \tag{1.4}$$

is referred to as the rank of the partially invariant manifold $\mathcal{M}$ with respect to the group G_r. Pair of integers (ρ, δ) define the type of the partially invariant manifold $\mathcal{M}$.

The rank of the manifold is equal to the dimension of the orbit $\mathcal{O}(\mathcal{M})$ in space of invariants of the group G_r. In practical calculations formula (1.4) is inconvenient, because it relies on the invariant representation (1.3) of the orbit of the manifold $\mathcal{M}$, which may not be known explicitly. However, by using (1.2), the rank can be found in terms of codimension s of the initial manifold $\mathcal{M}$ and its defect $\delta(\mathcal{M}, G_r)$,

$$\rho(\mathcal{M}, G_r) = \delta(\mathcal{M}, G_r) + t - s. \tag{1.5}$$

There is a convenient formula for the defect of a partially invariant manifold [1].

Theorem 1. *Let a partially invariant manifold $\mathcal{M}$ of the Lie group G_r be regularly defined by relations (1.1). Let $\{X_1, \ldots, X_r\}$ be the basis of infinitesimal generators of the group G_r. Then the defect of the partially invariant manifold $\mathcal{M}$ can be calculated by the formula*

$$\delta(\mathcal{M}, G_r) = \operatorname{rank}\|X_\alpha \psi^\sigma(x)|_{\mathcal{M}}\|. \tag{1.6}$$

The right-hand side of (1.6) represents the maximal rank of the matrix with elements $X_\alpha \psi^\sigma(x)$, calculated at points of the manifold $\mathcal{M}$.

Example 1. Let Lie group G_2 of transformations of four-dimensional space $\mathbb{R}^4(x, y, u, v)$ be generated by infinitesimal operators

$$X_1 = y\partial_x - x\partial_y, \qquad X_2 = v\partial_u - u\partial_v. \tag{1.7}$$

Here $n = 4$, $r = 2$. The complete set of functionally independent invariants of G_2 is

$$I^1 = x^2 + y^2, \qquad I^2 = u^2 + v^2. \tag{1.8}$$

Hence, $t = 2$ and the nonsingular G_2-invariant manifold can have rank either 1 or 0. Every non-singular G_2-invariant manifold of rank $\rho = 1$ can be locally written as

$$\mathcal{M}_1 : \qquad \Phi(I^1, I^2) = 0$$

with suitable function Φ. Manifold $\mathcal{M}_1$ represents the three-dimensional orbit of some G_2-partially invariant manifold. According to (1.6), any two-dimensional G_2-partially invariant manifold $\mathcal{M}_1^1$ can be obtained from $\mathcal{M}_1$ by addition of a non-invariant relation

$$\mathcal{M}_1^1 : \Phi(I^1, I^2) = 0, \qquad F(x, y, u, v) = 0.$$

Here functions F, I^1 and I^2 are functionally independent. Manifold $\mathcal{M}_1^1$ has rank $\rho = 1$ and defect $\delta = 1$. In the similar manner, addition of two non-invariant relations provides one with the general form of the one-dimensional partially invariant manifold of rank 1 and defect 2,

$$\mathcal{M}_1^2 : \Phi(I^1, I^2) = 0, \qquad F^1(x, y, u, v) = 0, \qquad F^2(x, y, u, v) = 0.$$

Similar construction of partially invariant manifolds of rank 0 is straightforward.

2. Partially invariant solutions

The definition of a partially invariant solution (PIS) as a natural generalization of an invariant solution (IS) of a differential equation was first suggested by L V Ovsiannikov [1, 7]. At present there are known many examples of partially invariant solutions, mostly for models of fluid mechanics [8–22]. Algorithm of construction of partially invariant solution is briefly recounted below.

Let us observe a system of differential equations

$$E : F^\sigma(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) = 0, \qquad \sigma = 1, \ldots, s. \tag{2.1}$$

The main space is $Z = \mathbb{R}^n(x) \times \mathbb{R}^m(u)$. By $\underset{p}{u}$ we denote the set of all pth order derivatives: $\left\{\frac{\partial^p u^j}{\partial x^{i_1} \ldots \partial x^{i_p}}\right\}$. System (2.1) admits Lie group $G_r = \{T_a : Z \times \mathbb{R}^r \to Z\}$. Action of G_r can be prolonged on the derivatives in usual manner [1, 2]. Let Lie algebra of infinitesimal generators of G_r be

$$L_r = \{X_\alpha = \xi^i_\alpha(x,u)\partial_{x^i} + \eta^k_\alpha(x,u)\partial_{u^k}, \alpha = 1,\ldots,r\}.$$

For a k-dimensional subalgebra $H \subset L_r$ the matrix

$$H(\xi) = \begin{pmatrix} \xi^1_1(x,u) & \ldots & \xi^n_1(x,u) \\ \ldots & \ldots & \ldots \\ \xi^1_k(x,u) & \ldots & \xi^n_k(x,u) \end{pmatrix}$$

and the extended matrix

$$H(\xi,\eta) = \begin{pmatrix} \xi^1_1(x,u) & \ldots & \xi^n_1(x,u) & \eta^1_1(x,u) & \ldots & \eta^m_1(x,u) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \xi^1_k(x,u) & \ldots & \xi^n_k(x,u) & \eta^1_k(x,u) & \ldots & \eta^m_k(x,u) \end{pmatrix}$$

are introduced. Let the manifold

$$U : u^i = \varphi^i(x), \qquad i = 1,\ldots,m \tag{2.2}$$

be a solution of equations (2.1).

Definition 5. *Solution U is referred to as H-invariant solution (H-IS) of the system of equations E if the manifold $U \subset Z$ is the invariant manifold under the subgroup $H \subset G_r$ action.*

The necessary condition of H-IS existence is convenient to formulate in terms of the corresponding Lie subalgebra of infinetesimal generators.

Lemma 1. *Lie subalgebra $H \subset L_r$ generates H-invariant solution of the system E if the following equality holds:*

$$\operatorname{rank} H(\xi) = \operatorname{rank} H(\xi,\eta). \tag{2.3}$$

Generalization of the notion of the invariant solution leads to the following.

Definition 6. *Solution U is called H-partially invariant solution (H-PIS) of the system E if the manifold $U \subset Z$ is the partially invariant manifold under $H \subset G_r$ action.*

PISs are usually constructed on subalgebras of the admissible algebra which do not satisfy the necessary condition (2.3). Definitions of rank and defect of a partially invariant manifold are transferred naturally on PISs. However, there is some specific owing to the distinction of roles of x and u variables.

Let us introduce the following integer characteristics of H:

$t = m + n - \operatorname{rank} H(\xi,\eta)$ —the total number of invariants of H;
$\sigma = n - \operatorname{rank} H(\xi)$ —the number of invariants of group H, which depend only on x;
$\mu = t - \sigma$ —the number of invariants essentially depending on u.

Partially invariant solution specified in the form of manifold Φ by formulae (2.2) has codimension $s = m$. The rank of manifold Φ can be calculated by formula (1.5) as $\rho = \delta + t - m$. The orbit of manifold Φ is an invariant manifold of group H; therefore, it may be specified by some functional relations of the form (1.3) for the invariants of the group. For the sake of simplicity we assume that the invariants are chosen in the separated form

$$I : \begin{cases} I^1(x,u),\ldots,I^\mu(x,u), \\ \lambda^1 = I^{t-\sigma+1}(x),\ldots,\lambda^\sigma = I^t(x). \end{cases} \tag{2.4}$$

At that, the full-rank condition for the Jacoby matrix is satisfied,

$$\operatorname{rank} \frac{\partial(I^1(x,u),\ldots,I^\mu(x,u))}{\partial(u^1,\ldots,u^m)} = \mu. \tag{2.5}$$

Let us construct the equations of the orbit of a partially invariant manifold Φ under the group H action. First, it is required to specify the rank of a partially invariant manifold. It can be taken as any integer ρ, satisfying the inequality

$$\sigma \leqslant \rho < \min\{n, t\}. \tag{2.6}$$

Definition 7. *Partially invariant solution is called regular if* $\rho = \sigma$.

The equations of the orbit $\mathcal{O}(\Phi, H)$ are constructed as the following set of functional relations between the invariants (2.4) of the group H,

$$\Phi^\tau(\lambda^1,\ldots,\lambda^\sigma, I^1,\ldots,I^\mu) = 0, \qquad \tau = 1,\ldots,t-\rho; \qquad \operatorname{rank}\|\partial\Phi^\sigma/\partial I^k\| = t-\rho \tag{2.7}$$

with unknown functions Φ^τ. In practical calculations these relations can be taken in the resolved form, e.g.

$$I^\tau = \varphi^\tau(\lambda^1,\ldots,\lambda^\sigma, I^{t-\rho+1},\ldots,I^\mu), \qquad \tau = 1,\ldots,t-\rho. \tag{2.8}$$

Although the resolved form is more convenient in practical PIS computations, it is not unique in the case of irregular solutions (i.e., $\rho > \sigma$). Hence, in the theoretical analysis it is preferable to refer to the general form (2.7).

The dimension of the manifold (2.7) in the space of invariants is ρ. Equations (2.7) can be solved with respect to $t-\rho$ functions u by virtue of conditions (2.5), (2.6). Without loss of generality one can assume these functions to be $u^1,\ldots,u^{t-\rho}$. The remaining $\delta = m-t+\rho$ functions $u^{t-\rho+1},\ldots,u^m$ do not have representation in terms of invariants of H and initially are not restricted by any extra assumptions. Thus, there appear δ non-invariant functions, which are assumed to depend arbitrarily on x,

$$u^{t-\rho+1} = w^1(x),\ldots,u^m = w^\delta(x). \tag{2.9}$$

Formulae (2.7), (2.9) define the representation of the partially invariant solution of the type (ρ,δ).

Remark 1. In practical calculations it is necessary to observe all non-equivalent possibilities for solution of the orbit equations (2.7) with respect to functions u. In what follows the representation of solution will refer to the combination of the orbit equations (2.7) with all possible representations of non-invariant functions of the form (2.9).

The substitution of the representation (2.7), (2.9) of solution into the system of equations (2.1) leads to a factor system of differential equations for the invariant functions $\Phi^k, k = 1,\ldots,\mu$, and non-invariant functions $w^j, j = 1,\ldots,\delta$. The factor system of a partially invariant solution contains a subsystem E/H for invariant functions and invariant variables, and equations Π for the non-invariant functions. System Π of equations should be observed as an overdetermined system for non-invariant functions w^j. At that, all invariant functions Φ^k are assumed to be known from solution of the invariant subsystem E/H. The compatibility conditions of Π usually extend both the invariant part E/H, and the system Π itself. The purpose of investigation at this stage is to bring system Π to involution, i.e. to obtain all of its compatibility conditions to prove its self-consistency. Unfortunately, it is impossible to trace this process in general form. If this step is performed, the factor system finally takes the form of a union of a subsystem E/H for the invariant functions and of compatible on the solutions of E/H system Π for determination of non-invariant functions. This reduced system is simpler than the original system E because E/H involves less independent variables, and Π contains less unknown functions.

Definition 8. *The union of the factor system E/H and of the system Π is referred to as H-partially invariant submodel of the system of differential equations E.*

Example 2. We continue the observations of example 1. Suppose that x and y are independent variables and u and v are sought functions in some system E of differential equations. One need to check, whether Lie group G_2 given by its infinitesimal generators (1.7) can generate invariant or partially invariant solution of equations E. Matrix $G_2(\xi, \eta)$ has the form

$$G_2(\xi, \eta) = \begin{pmatrix} y & -x & 0 & 0 \\ 0 & 0 & v & -u \end{pmatrix}.$$

Two first columns of this matrix give the matrix $G_2(\xi)$. One obtains $\operatorname{rank} G_2(\xi) = 1 < \operatorname{rank} G_2(\xi, \eta) = 2$, hence, group G_2 does not generate the invariant solution of equations E. This also follows from expressions of invariants (1.8), as it is impossible to express both unknown functions u and v in terms of invariants I^1 and I^2.

Next, the numerical characteristics of the group G_2 are $t = 2, \sigma = 1, \mu = 1$. According to (2.6) rank of the PIS can only be $\rho = 1$. Hence, the PIS is regular. The invariant part of the representation of solution (2.8) is

$$\sqrt{u^2 + v^2} = U(\sqrt{x^2 + y^2}).$$

The non-invariant part is either

$$\arctan(v/u) = \phi(x, y)$$

or

$$\arctan(u/v) = \phi(x, y)$$

depending on whether u or v vanishes in the domain of the solution. Substitution of this representation of the solution into equations E gives overdetermined system of two equations for function ϕ. Its compatibility condition yields the restriction for the invariant function U.

3. Partially invariant solutions hierarchy

Let $H, N \subset G_r$ be subgroups, such that H is a normal divisor in N: $H \lhd N$. Suppose that H does not satisfy conditions (2.3) of invariant solution existence. Let H-partially invariant submodel of E be known. In the sequel the following question is investigated: under which conditions on group N there exists N/H-invariant solution for H-PIS and how does it relate to N-PIS of equations E?

Owing to the one-to-one correspondence between local Lie groups of transformations and their Lie algebras of infinitesimal generators [1, 2], later on we do not distinguish between these two objects, denoting them by the same letter. All facts below proved in terms of the Lie group language can be translated into the Lie algebras language and *vice versa*.

Lemma 2. *The factor group N/H has induced action in the space of invariants of the group H.*

Proof. Factor group N/H is a set of all left equivalence classes $g_l = g \circ H = \{g \circ h | h \in H\}$. Let J be an invariant of the group H action. Action of g_l on J is defined as $g_l(J) = g \circ h(J) =$

$g(J)$, and, obviously, does not depend on the choice of the representative h of the equivalence class. According to the condition $H \lhd N$ one has $g^{-1}Hg \subset H$ for each $g \in N$. Let us show that if J is invariant with respect to H then for every $g \in N$ function $g(J)$ is also invariant under the group H action. Indeed, according to $g^{-1} \circ h \circ g(J) = h_1(J) = J$ we have $h(g(J)) = g(J)$ for each $h \in H$. Thus, the factor group N/H action is defined and closed in the space of invariants of group H. □

Lemma 3. *Group N/H is admitted by the system of differential equations E/H; group N is admitted by the system Π on solutions of E/H.*

Proof. Let us show that normal extension of group H acts on the set of H-invariant manifolds. Indeed, let $H \lhd N$ and $\mathcal{M}$ is some H-invariant manifold. For any $h \in H$ and $g \in N$ there exists $h_1 \in H$ such that $h \circ g = g \circ h_1$. Hence,

$$h(g(\mathcal{M})) = g(h_1(\mathcal{M})) = g(\mathcal{M}).$$

Thus, the transformed manifold $g(\mathcal{M})$ is also H-invariant.

The orbit of H-PIS, given by equations (2.7), is an invariant manifold of the group H. Then, any transformation of group N translates an orbit of H-PIS into an orbit of H-PIS. The orbit of an arbitrary H-PIS is determined by the system E/H. Action of the factor group N/H is closed in the space of invariants of group H. Hence, transformations of factor group N/H act on the set of solutions of E/H system, i.e., are admitted by this system of differential equations. Next, group N acts on the set of solutions of the original system N. Besides, it conserves the orbit of the solution in class of H-PISs. Thus, N acts in the set of H-PISs, i.e. is admitted by system Π on solutions of system E/H. □

The necessary condition of N/H-invariant solution existence is convenient to formulate in the Lie algebraic language using infinitesimal generators of the observed Lie group.

Lemma 4. *For N/H-invariant solution of the factor system E/H to exist the following condition should be satisfied:*

$$\operatorname{rank} N(\xi, \eta) - \operatorname{rank} N(\xi) = \operatorname{rank} H(\xi, \eta) - \operatorname{rank} H(\xi). \tag{3.1}$$

Proof. Let us transform infinitesimal generators of H into the coordinate system, which flattens the group action:

$$\begin{aligned} &y^1 = \lambda^1(x), \ldots, y^\sigma = \lambda^\sigma(x); \qquad && y^{\sigma+1} = x^{\sigma+1}, \ldots, y^n = x^n, \\ &v^1 = I^1(x,u), \ldots, v^\mu = I^\mu(x,u); \qquad && v^{\mu+1} = u^{\mu+1}, \ldots, v^m = u^m. \end{aligned} \tag{3.2}$$

Without loss of generality, this transformation is non-degenerate. Group H acts transitively in the space of variables $(y^{\sigma+1}, \ldots, y^n; v^{\mu+1}, \ldots, v^m)$. Representatives of the factor algebra N/H have the form $\bar{X} = X + Y$, where $Y \in H$. Operator Y does not contain any differentiations with respect to invariant variables $y^1, \ldots, y^\sigma, v^1, \ldots, v^\mu$. Operator X by virtue of lemma 2 projects into the space of invariants of group H, i.e. its coefficients at the differentiations with respect to invariant variables do not contain non-invariant variables. Thus, operator Y does not participate in projection of operator $\bar{X}$ into the space of invariants of group H. The Lie algebra of projections $\{X\}$ corresponds to the induced action of the factor group N/H in space of invariants of H. This Lie algebra should satisfy the necessary condition of the invariant solution existence (2.3). In what follows we check this condition explicitly in terms of coordinates of infinitesimal operators of Lie group N.

Matrix of coordinates of infinitesimal generators of Lie algebra N in (y, v) coordinates has the block structure

$$N(\xi,\eta) = \left(\begin{array}{c|c||c|c} 0 & A & 0 & B \\ 0 & 0 & 0 & C \\ \hline K & L & R & S \end{array}\right).$$

First two columns of blocks (to the left of the vertical double line) correspond to differentiations with respect to independent variables y, the remaining two columns (to the right of the double line) are coordinates at differentiations with respect to dependent variables v. A number of columns in each block separated by vertical lines is equal to σ, $n-\sigma$, μ, and $m-\mu$ correspondingly. The matrix horizontal division is such that first $\dim H$ rows (above the horizontal line) contain basic operators of Lie algebra H. The remaining operators (below the horizontal line) complete H to N.

Blocks of matrix $N(\xi,\eta)$ with coordinates (row, column) = $(1, 1)$, $(2, 1)$, $(1, 3)$ and $(2, 3)$ are equal to zero because coordinates at differentiations with respect to invariant variables in infinitesimal generators of H vanish. Algebra H does not satisfy the necessary conditions of existence of an invariant solution (2.3). Hence, by performing a suitable combination of rows above the horizontal line in matrix $N(\xi,\eta)$, one can make the block with coordinates $(2, 2)$ to be zero with $\operatorname{rank} A = n-\sigma$. As long as only the ranks of blocks are of interest, the linear combinations may be taken with coefficients depending on all variables. The resulting block C is also non-degenerate: $\operatorname{rank} C = m-\mu$. Thus, blocks A and C are square non-degenerate matrices of dimensions $(n-\sigma)\times(n-\sigma)$ and $(m-\mu)\times(m-\mu)$ respectively.

Let us turn to the rows of $N(\xi,\eta)$ placed below the horizontal line and corresponding to basic elements of complement of H to N. By a non-degenerate combination of these rows with upper $n-\sigma$ rows of matrix $N(\xi,\eta)$, one can zero block L. Next, by virtue of the non-degeneracy of block C, one can zero block S. At that, blocks K and R remain unchanged. Differential operators with coordinates from blocks K and R acting in the space of invariants of group H are the sought infinitesimal operators of induced action of the factor group N/H in the space of invariants. The necessary condition of the existence of invariant N/H-solution (2.3) is that the rank of block K is equal to the rank of matrix composed from blocks K and R,

$$\operatorname{rank} K = \operatorname{rank}(K, R). \tag{3.3}$$

This condition can be reformulated in terms of ranks of blocks of the complete matrix $N(\xi,\eta)$. Let columns of blocks of matrix $N(\xi,\eta)$ be denoted by Roman numbers I–IV. Note, that by virtue of non-degeneracy of blocks A and C columns II and IV are linearly independent with each other and with columns I and III.

$$\operatorname{rank}(\mathrm{I},\mathrm{II}) = \operatorname{rank} K + n - \sigma,$$

$$\operatorname{rank}(\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV}) = \operatorname{rank}(K,R) + n - \sigma + m - \mu.$$

Subtraction of the first equality from the second one gives

$$\operatorname{rank}(\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV}) - \operatorname{rank}(\mathrm{I},\mathrm{II}) = \operatorname{rank}(K,R) - \operatorname{rank} K + m - \mu.$$

Relation (3.3) is satisfied if and only if

$$\operatorname{rank}(\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV}) - \operatorname{rank}(\mathrm{I},\mathrm{II}) = m - \mu,$$

which is equivalent to (3.1) condition. As long as change of variables (3.2) does not affect ranks of matrices, this condition can be verified in initial coordinate system. □

Remark 2. The necessary conditions formulated in lemmas 1 and 4 guarantee the possibility of construction of representations of the corresponding invariant solutions and finding factor systems of equations for invariant functions. These conditions are not sufficient for the solution existence because compatibility of the obtained factor systems cannot be proved *a priory*.

4. Two-step construction of the solution

The information obtained about partially invariant solutions allows formulating the following statement.

Theorem 2. *Let the system of differential equations E admits Lie group of continuous transformations N. Suppose that there is a normal divisor H in N, which does not satisfy the condition of the existence of the invariant solution (2.5), but fulfils the requirement (3.1). Then, for the factor system E/H, corresponding to H-PIS there exists an invariant solution with respect to the factor group N/H. Moreover, the factor system of the invariant solution $(E/H)/(N/H)$ is equivalent to the factor system of the partially invariant solution E/N.*

Proof. The possibility of construction of N/H-invariant solution of H-PIS is already shown in lemmas 2–4. The only thing left to demonstrate is the equivalence of factor systems E/N obtained directly and by using the two-step method as $(E/H)/(N/H)$.

Suppose, that the dimension of the factor group N/H is equal to κ. Owing to condition (3.3) one can assume invariants $\lambda^1, \ldots, \lambda^\sigma$, and $I^1, \ldots, I^\mu$ of the group H to be chosen in such a way that functions

$$\lambda^1, \ldots, \lambda^{\sigma-\kappa}, \qquad I^1, \ldots, I^\mu \tag{4.1}$$

form the basis of functionally independent invariants of the group N. This implies that the action of the factor group N/H is transitive in the subspace $\mathbb{R}^\kappa(\lambda^{\sigma-\kappa+1}, \ldots, \lambda^\sigma)$, whereas variables (4.1) are independent invariants of N/H. The representation of N/H-invariant solution of the factor system E/H is obtained by the demand of independency of functions Φ in (2.7) on variables $\lambda^{\sigma-\kappa+1}, \ldots, \lambda^\sigma$. Exactly, the same representation of the orbit of N-PIS of equations E is obtained directly. Thus, representations of N-PIS and of $(E/H)/(N/H)$-IS coincide, hence their factor systems are equivalent as well. □

Thus, in the set of partially invariant solutions of investigated system there is a hierarchic structure.

Definition 9. *A partially invariant submodel is called indecomposable if it cannot be represented as the non-trivial combination of a partially invariant and invariant submodels.*

Investigation of only indecomposable submodels allows significant reduction of efforts in enumeration of all partially invariant solutions of a given system of differential equations. Indeed, the most labour-intensive step of involutivity analysis for overdetermined systems of differential equations should be done only for indecomposable submodels. The remaining submodels are obtained from the indecomposable ones by means of only invariant reductions, which is usually much simpler.

Let us assume a system E of differential equations with known admitted Lie algebra L_r and known optimal system of subalgebras ΘL_r. In order to obtain the complete list of subalgebras, which generate essentially different indecomposable PISs one need to look through all subalgebras from ΘL_r to find out whether each subalgebra $N \in \Theta L_r$ contains an ideal $H \subset N$ such that condition (3.1) holds. If such ideal H exists, then the subalgebra generates decomposable PIS and should be omitted. Otherwise, it generates indecomposable PIS, which can be obtained according to the algorithm given in section 2. Note, that this construction essentially depends on the particular representation the abstract Lie algebra L_r as the Lie algebra of infinitesimal generators of the Lie group G_r of transformations, admitted by the system E.

In practical calculations it is useful to know which indecomposable PIS of higher rank contains a given decomposable N-submodel. Suppose, that there exists an ideal $H \subset N$ satisfying condition (3.1). Optimal system of subalgebras ΘL_r necessary contains a subalgebra H', which is equivalent to H by means of action of inner automorphisms of L_r. Thus, according to theorem 2, N-PIS can be obtained as an N'/H'-invariant submodel of H'-PIS.

5. Shallow water equations

The equations, describing motions of a thin water layer over a flat bottom are observed:

$$E: \quad \begin{aligned} &u_t + uu_x + vu_y + h_x = 0,\\ &v_t + uv_x + vv_y + h_y = 0,\\ &h_t + (uh)_x + (vh)_y = 0. \end{aligned} \tag{5.1}$$

Here (u, v) is a particle's velocity vector, h is the depth of the water layer. The basic space here is $\mathbb{R}^3(t, x, y) \times \mathbb{R}^3(u, v, h)$, hence $n = m = 3$. The admissible algebra L_9 [7] is generated by operators (notations of paper [24] are adopted):

$$\begin{aligned} &X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_4 = t\partial_x + \partial_u, \qquad X_5 = t\partial_y + \partial_v,\\ &X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u, \qquad X_{10} = \partial_t,\\ &X_{11} = x\partial_x + y\partial_y + u\partial_u + v\partial_v + 2h\partial_h,\\ &X_{12} = t^2\partial_t + tx\partial_x + ty\partial_y + (x - tu)\partial_u + (y - tv)\partial_v - 2th\partial_h.\\ &X_{13} = 2t\partial_t + x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2h\partial_h. \end{aligned}$$

Let us observe a partially invariant solution given by Lie subalgebra

$$N = \{X_1, X_4, X_{10} + X_{12}\} \subset L_9.$$

The subalgebra $H = \{X_1, X_4\}$ in N is selected. It is easy to check that H is ideal in N. Condition (3.1) is satisfied, therefore one can apply the two-step algorithm.

The complete set of functionally independent invariants of H is

$$t, y, v, h.$$

Here $n = 3, m = 3, t = 4, \sigma = 2, \mu = 2$. Let us construct an indecomposable H-PIS of rank 2. The equation of orbit of a partially invariant solution (2.7) can be written in an explicit form

$$v = v(t, y), \qquad h = h(t, y).$$

The defect of the solution is $\delta = 1$. There is only one non-invariant function u, which is supposed to depend on all independent variables,

$$u = u(t, x, y).$$

Substitution of the obtained representation of solution into the initial system (5.1) gives the submodel equations. The first and the third equations of system (5.1) form an overdetermined system Π for the non-invariant function u. From the third equation of (5.1) it follows that u is linear with respect to x,

$$u = k(t, y)x + U(t, y). \tag{5.2}$$

At that, function k has an expression in terms of invariant functions: $k = -(h_t + vh_y)/h$. For the sake of convenience, one can treat this expression as an additional equation of the factor system E/H. The substitution of the representation (5.2) into equations (5.1) and splitting with respect to x leads to the system for invariant functions:

$$\begin{aligned} &v_t + vv_y + h_y = 0,\\ &h_t + vh_y + kh = 0,\\ &k_t + vk_y + k^2 = 0 \end{aligned} \tag{5.3}$$

and to equation for function U,

$$U_t + vU_y + Uk = 0. \tag{5.4}$$

Equations (5.3) form the factor system E/H, whereas equations (5.2), (5.4) represent trivially consistent system Π for the non-invariant function. The factor system (5.3) itself admits some Lie group symmetries. According to lemma 3 the admissible group contains the subgroup with Lie algebra

$$\frac{\mathrm{Nor}_{L_9}\{X_1, X_4\}}{\{X_1, X_4\}} = \{X_2, X_5, X_{10}, X_{11}, X_{12}, X_{13}\}.$$

In particular, this algebra contains the subalgebra $N/H = \{X_{10} + X_{12}\}$. For construction of N/H-invariant solution of the H-PIS, operator $X_{10} + X_{12}$ should be re-written in terms of invariants of algebra H in the following form (for convenience, it is also prolonged to the invariant variable k)

$$(t^2+1)\partial_t + ty\partial_y + (y - tv)\partial_v - 2th\partial_h + (1 - 2tk)\partial_k.$$

Invariants of this operator are

$$\lambda = y/\sqrt{t^2+1}, \qquad V = v\sqrt{t^2+1} - t\lambda, \qquad H = h(t^2+1), \qquad K = k(t^2+1) - t.$$

Necessary condition of an invariant solution existence is obviously satisfied. Representation of the invariant solution of the factor system (5.3) has the form

$$v = \frac{V(\lambda) + t\lambda}{\sqrt{t^2+1}}, \qquad h = \frac{H(\lambda)}{t^2+1}, \qquad k = \frac{K(\lambda) + t}{t^2+1}. \tag{5.5}$$

Substitution into equations (5.3) gives

$$\begin{aligned} &VV' + H' = -\lambda,\\ &VK' + K^2 + 1 = 0,\\ &VH' + HV' = -KH. \end{aligned} \tag{5.6}$$

Equations (5.6) form a factor system $(E/H)/(N/H) = E/N$ for the N-PIS of equations E. The corresponding system Π is given by expression (5.2) and by equation (5.4) with substitution of invariant functions (5.5). For the integration of the obtained system new independent variable μ is introduced,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = V(\lambda), \qquad \mu = \int \frac{\mathrm{d}\lambda}{V(\lambda)}. \tag{5.7}$$

Then, the second equation of (5.6) accurate to insufficient constant yields

$$K = -\tan\mu. \tag{5.8}$$

By using (5.8), one can integrate equation (5.4),

$$U = \frac{f(\mu - \arctan t)}{\cos\mu\sqrt{t^2+1}} \tag{5.9}$$

(f is an arbitrary function). Besides, system (5.6) has a first integral, which follows from its third equation

$$HV\cos\mu = m. \tag{5.10}$$

Finally, system (5.6) possesses a Bernoulli integral

$$V^2 + \lambda^2 + 2H = b^2, \qquad b = \text{const}. \tag{5.11}$$

The latter should be observed as an implicit (not resolved with respect to the derivative) equation for the dependence $\lambda(\mu)$,

$$(\lambda')^2 + \lambda^2 + \frac{2m}{\lambda' \cos\mu} = b^2. \tag{5.12}$$

Hence, the N-PIS is finally given by expressions (5.2), (5.5), where functions V, H, K and U can be found form (5.7)–(5.11) after integration of the first-order ODE (5.12). Note, that the solution contains an arbitrary function f. Further analysis of the physical properties of this solution can be found in [30].

Analysis of the optimal system [24] for the nine-dimensional Lie algebra L_9 [7], admitted by equations (5.1) shows that the combination of operators $\{\partial_x, t\partial_x + \partial_u\}$ or the equivalent combination $\{\partial_y, t\partial_y + \partial_v\}$ is presented in nine three-dimensional representatives. All PISs of defect 1 generated by these subalgebras are decomposable and can be obtained by invariant reduction of equations (5.3) by means of one of the following operators:

$$\begin{gathered} X_2,\, X_{11},\, X_{10} + X_{11},\, X_5 + X_{10},\, X_{10},\, aX_{11} + X_{13},\, X_5 + X_{11} + X_{13}, \\ aX_{11} + X_{10} + X_{12}, \end{gathered} \tag{5.13}$$

Here a is an arbitrary real parameter.

6. MHD with general state equation

Equations of ideal magnetohydrodynamics [25, 26] are observed:

$$\begin{aligned} &D\rho + \rho \operatorname{div} \mathbf{u} = 0, \\ &D\mathbf{u} + \rho^{-1}\nabla p + \rho^{-1}\mathbf{H} \times \operatorname{rot} \mathbf{H} = 0, \\ &Dp + A(p, \rho) \operatorname{div} \mathbf{u} = 0, \\ &D\mathbf{H} + \mathbf{H} \operatorname{div} \mathbf{u} - (\mathbf{H} \cdot \nabla)\mathbf{u} = 0, \\ &\operatorname{div} \mathbf{H} = 0, \qquad D = \partial_t + \mathbf{u} \cdot \nabla. \end{aligned}$$

Here $\mathbf{u} = (u, v, w)$ is a velocity vector, $\mathbf{H} = (H, K, L)$ is the magnetic field; p and ρ are pressure and density. Thermodynamical functions are related by the state equation $p = F(S, \rho)$ with entropy S. Function $A(p, \rho)$ is determined by the state equation as $A = \rho(\partial F/\partial \rho)$. All functions depend on time t and Cartesian coordinates $\mathbf{x} = (x, y, z)$.

The admissible group is 11-dimensional Galilean group extended by homothety [10, 27]. Infinitesimal operators form Lie algebra L_{11} with basis

$$\begin{aligned} &X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = \partial_z, \qquad X_4 = t\partial_x + \partial_u, \qquad X_5 = t\partial_y + \partial_v, \\ &X_6 = t\partial_z + \partial_w, \qquad X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v + K\partial_L - L\partial_K, \\ &X_8 = z\partial_x - x\partial_z + w\partial_u - u\partial_w + L\partial_H - H\partial_L, \\ &X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u + H\partial_K - K\partial_H, \\ &X_{10} = \partial_t, \qquad X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z. \end{aligned}$$

Optimal system of subalgebras ΘL_{11} was constructed in [3, 28]; in the final form it can be found in [29]. Regular PISs for gas dynamics equations generated by representatives of this optimal system were investigated in papers [13–16]. By virtue of special form of operators of Lie algebra L_{11}, partially invariant solutions of MHD are generated by the same representatives of the optimal system as in pure gas dynamics without the magnetic field.

Hence, the known results can be taken into account in construction of the solutions for MHD equations. Only non-barochronous solutions will be observed below, i.e. solutions in which pressure p depends on spatial variables $\mathbf{x}$. Barochronous (pressure depends only on time) motions of ideal compressible fluid were carefully observed in general form in [18–20]. For ideal MHD the similar result is not yet known, however, it is reasonable to distinguish the barochronous motions of plasma into a separate class.

Investigation of the optimal system of subalgebras for the Lie algebra L_{11} reveals more than 40 subalgebras, which generate regular PISs of MHD equations. Classification of this large class of solutions is done below. As investigation of each of partially invariant submodel requires additional efforts, our goal here is only to classify the set of regular partially invariant solutions to MHD equations. Construction and physical interpretation of the solutions will be done in a separate paper.

All operators of Lie algebra L_{11} have nonzero coordinates at differentiations with respect to independent variables, hence, one-dimensional subalgebras do not give rise to PISs. The only two-dimensional subalgebra, which generates PIS of defect 1 and rank 3 is $\{X_1, X_4\}$.

Analysis of three-dimensional representatives of the optimal system shows that there are 15 subalgebras, responsible for PISs of defect 1 and rank 2. All of these solutions are generated by indecomposable PISs constructed on the following subalgebras:

$$\begin{gathered}\{X_2, X_3, X_7\}, \qquad \{X_5, X_6, X_7\}, \qquad \{X_7, X_8, X_9\}, \\ \{X_3 + X_5, X_2 - X_6, X_7\}, \qquad \{X_3, X_5, X_2 + X_6\}.\end{gathered} \tag{6.1}$$

All the rest of submodels of defect 1 and rank 2 can be obtained by the invariant reduction of $\{X_1, X_4\}$-PIS with respect to one of the following operators:

$$\begin{gathered}X_7 + aX_{11}, \qquad X_7 + X_{10}, \qquad aX_6 + X_{11}, \\ X_5 + X_{10}, \qquad X_{10}, \qquad X_2 + X_6, \qquad X_6, \qquad X_2.\end{gathered}$$

There are 31 four-dimensional representatives of the optimal system, which give rise to PISs of defect 1 and rank 1. These are representatives of ΘL_{11} with numbers 1, 4, 5 (at $\alpha = 0$), 6, 7 (at $\alpha = 0$), 9 (at $\beta = 0$), 10 (at $\alpha = 0$), 12–14, 16 (at $\alpha = 0$), 17–21, 23, 29, 30, 35, 36, 38, 41–46, 48, 49 (numeration is given according to [29]). There are only nine indecomposable PISs among them with bases

$$\begin{gathered}\{X_1, X_5, X_6, \alpha X_4 + X_7; \alpha \neq 0\}, \qquad \{\alpha X_1 + X_4, X_5, X_6, \beta X_1 + X_7; \beta \neq 0\}, \\ \{X_1, X_2, X_3, \alpha X_4 + X_7; \alpha \neq 0\}, \qquad \{\alpha X_1 + X_4, X_3 + X_5, X_2 - X_6, \beta X_1 + X_7; \beta \neq 0\} \\ \{X_2, X_3, X_4, X_1 + X_7\}, \qquad \{X_2, \alpha X_1 + X_3, X_1 + X_5, X_6; \alpha \neq 0\}, \\ \{X_1, X_3 + X_5, X_2 - X_6, \alpha X_4 + X_7; \alpha \neq 0\}, \qquad \{X_1, X_2, X_3 + X_5, X_6\}, \\ \{X_1, \alpha X_2 + \beta X_3 + X_4, \sigma X_3 + X_5, \tau X_2 + X_6; \alpha^2 + \beta^2 = 1, \alpha^2 + \tau^2 \neq 0, \beta^2 + \sigma^2 \neq 0\}.\end{gathered}$$

All these subalgebras generate barochronous submodels, since their only invariant independent variable is time t. The only non-barochronous indecomposable partially invariant submodel is generated by a four-dimensional subalgebra $\{X_2, X_3, X_5, X_6\}$. The latter gives PIS of defect 2 and rank 2.

Among five-dimensional subalgebras of L_{11} the only regular indecomposable and non-barochronous solution is generated by subalgebra $\{X_2, X_3, X_5, X_6, X_7\}$. For MHD equations this solution has defect 3 and rank 2. The remaining regular partially invariant solutions generated by higher-dimensional subalgebras of L_{11} are either barochronous or decomposable. The calculations above are summarized in the following theorem.

Theorem 3. *The class of indecomposable regular non-barochronous PISs for ideal MHD equations is exhausted by the submodels generated by subalgebra* $\{X_1, X_4\}$ *(defect 1, rank*

3), subalgebras (6.1) (defect 1, rank 2), subalgebra $\{X_2, X_3, X_5, X_6\}$ *(defect 2, rank 2) and subalgebra* $\{X_2, X_3, X_5, X_6, X_7\}$ *(defect 3, rank 2).*

Analysis of the enumerated submodels will be presented as a separate paper. Below we give only the brief description of these submodels.

Investigation of $\{X_1, X_4\}$-submodel is similar to the one given in section 5 for shallow water equations. The only non-invariant function is u. The invariant variables are t, y and z. From the continuity equation it follows that $u = xM(t, y, z) + U(t, y, z)$. The remaining function depends only on the invariant variables. After some integration, equations of the submodel are reduced to a compatible system of seven equations with three independent variables.

Partially invariant submodels generated by subalgebras $\{X_7, X_8, X_9\}$ and $\{X_2, X_3, X_7\}$ were studied in [31–34]. They can be treated as 3D generalizations of classical 1D solutions with planar or spherical waves. The difference with the classical solutions is that the velocity and magnetic field vectors have nonzero tangential to the wave front components, which depend on the position on the particle on the wave front. The orientation of the tangential components of the vectors is determined by a finite relation with functional arbitrariness. The construction of 3D picture of motion requires calculation of the particle trajectory and magnetic field line patterns. They are determined by the invariant subsystem of equations with two independent variables (time t and spatial coordinate r). The patterns are attached to planar or spherical wave fronts according to the relation for the tangential components of velocity and magnetic field vectors. Since the orientation of trajectories and magnetic lines in 3D space is determined with functional arbitrariness, it is possible to obtain infinitely many pictures of plasma motion with the same shapes of trajectories and magnetic field lines but with different positions of these curves in 3D space. This may be treated as the nonlinear superposition principle for trajectories and magnetic lines.

Subalgebras $\{X_5, X_6, X_7\}$ and $\{X_3 + X_5, X_2 - X_6, X_7\}$ generate solutions, which are similar to the $\{X_2, X_3, X_7\}$-solution, but with additional plasma propagation along planar wave fronts. The remaining rank 2 solution generated by the subalgebra $\{X_3, X_5, X_2 + X_6\}$ describes plasma motion, where the velocity component v depends linearly on y and z and all the remaining functions depend only on t and x.

Submodels with two non-invariant functions are more difficult for compatibility analysis. The partially invariant submodel generated by subalgebra $\{X_2, X_3, X_5, X_6\}$ describes a solution, where v and w depend on all independent variables, while the remaining functions depend only on t and x. The compatibility analysis reveals that v and w are linear on y and z. The resulting system of differential equations with two independent variables is brought to involution and partially integrated. Analysis of the defect 3 submodel generated by $\{X_2, X_3, X_5, X_6, X_7\}$ is not yet completed. Investigation of this submodel is close to the compatibility analysis for the general form of barochronous (pressure p or complete pressure $p + \mathbf{H}^2/2$ depends only on time) solutions of ideal MHD equations, which is also not finished yet.

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